

# GENERALIZED $\gamma$ -GENERATING MATRICES AND NEHARI-TAKAGI PROBLEM

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*To the Memory of Leiba Rodman*

ABSTRACT. Under certain mild assumption, we establish a one-to-one correspondence between solutions of the Nehari-Takagi problem and solutions of some Takagi-Sarason interpolation problem. The resolvent matrix of the Nehari-Takagi problem is shown to belong to the class of so-called generalized  $\gamma$ -generating matrices, which is introduced and studied in the paper.

## 1. INTRODUCTION

For a bounded function  $f$  defined on  $\mathbb{T} = \{z : |z| = 1\}$  let us set

$$(1.1) \quad \gamma_k(f) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{ik\theta} f(e^{i\theta}) d\theta \quad (k = 1, 2, \dots).$$

The Nehari problem consists of the following: given a sequence of complex numbers  $\gamma_k$  ( $k \in \mathbb{N}$ ) find a function  $f \in L_\infty(\mathbb{T})$  such that  $\|f\| \leq 1$  and

$$(1.2) \quad \gamma_k(f) = \gamma_k, \quad (k = 1, 2, \dots).$$

By Nehari theorem [22] this problem is solvable if and only if the Hankel matrix  $\Gamma = (\gamma_{i+j-1})_{i,j=1}^\infty$  determines a bounded operator in  $l_2(\mathbb{N})$  with  $\|\Gamma\| \leq 1$ . The problem (1.2) is called indeterminate if it has infinitely many solutions. A criterion for the Nehari problem to be indeterminate and a full description of the set of its solutions was given in [1], [2].

In [2] Adamyan, Arov and Krein considered the following indefinite version of the Nehari problem, so called Nehari-Takagi problem  $\mathbf{NTP}_\kappa(\Gamma)$ : Given  $\kappa \in \mathbb{N}$  and a sequence  $\{\gamma_k\}_{k=1}^\infty$  of complex numbers, find a function  $f \in L_\infty(\mathbb{T})$ , such that  $\|f\|_\infty \leq 1$  and

$$\text{rank}(\Gamma(f) - \Gamma) \leq \kappa.$$

Here  $\Gamma(f)$  is the Hankel matrix  $\Gamma(f) := (\gamma_{i+j-1}(f))_{i,j=1}^\infty$ . As was shown in [2], the problem  $\mathbf{NTP}_\kappa(\Gamma)$  is solvable if and only if the total multiplicity  $\nu_-(I - \Gamma^*\Gamma)$  of the negative spectrum of the operator  $I - \Gamma^*\Gamma$  does not exceed  $\kappa$ . In the case when the operator  $I - \Gamma^*\Gamma$  is invertible and  $\nu_-(I - \Gamma^*\Gamma) = \kappa$ , the set of solutions of this problem was parameterized by the formula

$$(1.3) \quad f(\mu) = (a_{11}(\mu)\varepsilon(\mu) + a_{12}(\mu))(a_{21}(\mu)\varepsilon(\mu) + a_{22}(\mu))^{-1},$$

where  $\mathfrak{A}(\mu) = (a_{ij}(\mu))_{i,j=1}^2$  is the so-called  $\gamma$ -generating matrix and the parameter  $\varepsilon$  ranges over the Schur class of functions bounded and holomorphic on  $\mathbb{D} = \{z :$

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$\|z\| < 1\}$ . In [2] applications of the Nehari-Takagi problem to various approximation and interpolation problems were presented. Matrix and operator versions of Nehari problem were considered in [25] and [3]. In the rational case matrix Nehari and Nehari-Takagi problems were studied in [10]. A complete exposition of these results can be found also in [24] and [7].

In the present paper we consider the general matrix Nehari-Takagi problem and show that under some assumptions this problem can be reduced to Takagi-Sarason interpolation problem studied earlier in [14]. Using the results from [14], [15] we obtain a description of the set of solutions of the matrix Nehari-Takagi problem in the form (1.3).

Connections between the class of generalized  $\gamma$ -generating matrices and the class of generalized  $j$ -inner matrix valued functions (mvf's) introduced in [13] is established. Using this connection we present another proof of the formula for the resolvent matrix  $\mathfrak{A}(\mu)$  from [10] in the case when the Hankel matrix  $\Gamma$  corresponds to a rational mvf. All the results, except the last section, are presented in unified notations both for the unit circle  $\mathbb{T}$  and the real line  $\mathbb{R}$ .

Now we briefly describe the content of the paper. In Section 2 some preliminary statements concerning generalized Schur mvf's and generalized  $j$ -inner mvf's are given. In Section 3 the Takagi-Sarason interpolation problem is studied. In Section 4 we introduce and study the class of generalized  $\gamma$ -generating matrices and establish their connection with the class of generalized  $j$ -inner mvf's. In Section 5 we consider Nehari-Takagi problem and under certain mild assumption, establish a one-to-one correspondence between solutions of the Nehari-Takagi problem and solutions of some Takagi-Sarason interpolation problem. In Section 6 we calculate the resolvent matrix of the Nehari-Takagi problem in the rational case and show that it coincides with the resolvent matrix found in [10].

## 2. PRELIMINARIES

**2.1. Notations.** Let  $\Omega_+$  be either  $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$  or  $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \text{Im } \lambda > 0\}$ . Let us set for arbitrary  $\lambda, \omega \in \mathbb{C}$

$$\rho_\omega(\lambda) = \begin{cases} 1 - \lambda\bar{\omega}, & \Omega_+ = \mathbb{D}, \\ -i(\lambda - \bar{\omega}), & \Omega_+ = \mathbb{C}_+, \end{cases} \quad \lambda^\circ = \begin{cases} 1/\bar{\lambda}, & \Omega_+ = \mathbb{D}, \\ \bar{\lambda}, & \Omega_+ = \mathbb{C}_+. \end{cases}$$

Thus,  $\Omega_+ = \{\omega \in \mathbb{C} : \rho_\omega(\omega) > 0\}$  and let

$$\Omega_0 = \{\omega \in \mathbb{C} : \rho_\omega(\omega) = 0\}, \quad \Omega_- = \{\omega \in \mathbb{C} : \rho_\omega(\omega) < 0\}.$$

The following basic classes of mvf's will be used in this paper:  $H_2^{p \times q}$  (resp.,  $H_\infty^{p \times q}$ ) is the class of  $p \times q$  mvf's with entries in the Hardy space  $H_2$  (resp.,  $H_\infty$ );  $H_2^p := H_2^{p \times 1}$ , and  $(H_2^p)^\perp = L_2^p \ominus H_2^p$ ,  $\mathcal{S}^{p \times q}$  is the Schur class of  $p \times q$  mvf's holomorphic and contractive on  $\Omega_+$ ,  $\mathcal{S}_{in}^{p \times q}$  (resp.,  $\mathcal{S}_{out}^{p \times q}$ ) is the class of inner (resp., outer) mvf's in  $\mathcal{S}^{p \times q}$ :

$$\begin{aligned} \mathcal{S}_{in}^{p \times q} &= \{s \in \mathcal{S}^{p \times q} : s(\mu)^* s(\mu) = I_p \text{ a.e. on } \Omega_0\}; \\ \mathcal{S}_{out}^{p \times q} &= \{s \in \mathcal{S}^{p \times q} : \overline{s H_2^q} = H_2^p\}, \end{aligned}$$

The Nevanlinna class  $\mathcal{N}^{p \times q}$  and the Smirnov class  $\mathcal{N}_+^{p \times q}$  are defined by

$$(2.1) \quad \begin{aligned} \mathcal{N}^{p \times q} &= \{f = h^{-1}g : g \in H_\infty^{p \times q}, h \in \mathcal{S}^{1 \times 1}\}, \\ \mathcal{N}_+^{p \times q} &= \{f = h^{-1}g : g \in H_\infty^{p \times q}, h \in \mathcal{S}_{out}^{1 \times 1}\}. \end{aligned}$$

For a mvf  $f(\lambda)$  let us set  $f^\#(\lambda) = f(\lambda^\circ)^*$ . Denote by  $\mathfrak{h}_f$  the domain of holomorphy of the mvf  $f$  and let  $\mathfrak{h}_f^\pm = \mathfrak{h}_f \cap \Omega_\pm$ .

A  $p \times q$  mvf  $f_-$  in  $\Omega_-$  is said to be a pseudocontinuation of a mvf  $f \in \mathcal{N}^{p \times q}$ , if

- (1)  $f_-^\# \in \mathcal{N}^{p \times q}$ ;
- (2)  $\lim_{\nu \downarrow 0} f_-(\mu - i\nu) = \lim_{\nu \downarrow 0} f_+(\mu + i\nu) (= f(\mu))$  a.e. on  $\Omega_0$ .

The subclass of all mvf's  $f \in \mathcal{N}^{p \times q}$  that admit pseudocontinuations  $f_-$  into  $\Omega_-$  will be denoted  $\Pi^{p \times q}$ .

Let  $\varphi(\lambda)$  be a  $p \times q$  mvf that is meromorphic on  $\Omega_+$  with a Laurent expansion

$$\varphi(\lambda) = (\lambda - \lambda_0)^{-k} \varphi_{-k} + \cdots + (\lambda - \lambda_0)^{-1} \varphi_{-1} + \varphi_0 + \cdots$$

in a neighborhood of its pole  $\lambda_0 \in \Omega_+$ . The pole multiplicity  $\mathcal{M}_\pi(\varphi, \lambda_0)$  is defined by (see [20])

$$\mathcal{M}_\pi(\varphi, \lambda_0) = \text{rank } L(\varphi, \lambda_0), \quad T(\varphi, \lambda_0) = \begin{bmatrix} \varphi_{-k} & & \mathbf{0} \\ \vdots & \ddots & \\ \varphi_{-1} & \cdots & \varphi_{-k} \end{bmatrix}.$$

The pole multiplicity of  $\varphi$  over  $\Omega_+$  is given by

$$\mathcal{M}_\pi(\varphi, \Omega_+) = \sum_{\lambda \in \Omega_+} \mathcal{M}_\pi(\varphi, \lambda).$$

This definition of pole multiplicity coincides with that based on the Smith-McMillan representation of  $\varphi$  (see [10]).

Let  $b_\omega(\lambda)$ , be a Blaschke factor ( $b_\omega(\lambda) = \frac{\lambda - \omega}{1 - \bar{\lambda}\omega^*}$  in the case  $\Omega_+ = \mathbb{D}$ ,  $b_\omega(\lambda) = \frac{\lambda - \omega}{\bar{\lambda} - \omega^*}$  in the case  $\Omega_+ = \mathbb{C}_+$ ), and let  $P$  be an orthogonal projection in  $\mathbb{C}^p$ . Then the mvf

$$B_\alpha(\lambda) = I_p - P + b_\alpha(\lambda)P, \quad \omega \in \Omega_+,$$

belongs to the Schur class  $\mathcal{S}^{p \times p}$  and is called *the elementary Blaschke-Potapov (BP) factor* and  $B(\lambda)$  is called *primary* if  $\text{rank } P = 1$ . The product

$$B(\lambda) = \prod_{j=1}^{\kappa} B_{\alpha_j}(\lambda),$$

where  $B_{\alpha_j}(\lambda)$  are primary Blaschke-Potapov factors is called *a Blaschke-Potapov product* of degree  $\kappa$ .

*Remark 2.1.* For a Blaschke-Potapov product  $b$  the following statements are equivalent:

- (1) the degree of  $b$  is equal  $\kappa$ ;
- (2)  $\mathcal{M}_\pi(b^{-1}, \Omega_+) = \kappa$ ;

**2.2. The generalized Schur class.** Let  $\kappa \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ . Recall, that a Hermitian kernel  $\mathbf{K}_\omega(\lambda) : \Omega \times \Omega \rightarrow \mathbb{C}^{m \times m}$  is said to have  $\kappa$  negative squares, if for every positive integer  $n$  and every choice of  $\omega_j \in \Omega$  and  $u_j \in \mathbb{C}^m$  ( $j = 1, \dots, n$ ) the matrix

$$(\langle \mathbf{K}_{\omega_j}(\omega_k) u_j, u_k \rangle)_{j,k=1}^n$$

has at most  $\kappa$  negative eigenvalues, and for some choice of  $\omega_1, \dots, \omega_n \in \Omega$  and  $u_1, \dots, u_n \in \mathbb{C}^m$  exactly  $\kappa$  negative eigenvalues (see [20]).

Let  $\mathcal{S}_\kappa^{q \times p}$  denote the *generalized Schur class* of  $q \times p$  mvf's  $s$  that are meromorphic in  $\Omega_+$  and for which the kernel

$$(2.2) \quad \Lambda_\omega^s(\lambda) = \frac{I_p - s(\lambda)s(\omega)^*}{\rho_\omega(\lambda)}$$

has  $\kappa$  negative squares on  $\mathfrak{h}_s^+ \times \mathfrak{h}_s^+$ . In the case where  $\kappa = 0$ , the class  $\mathcal{S}_0^{q \times p}$  coincides with the Schur class  $\mathcal{S}^{q \times p}$  of contractive mvf's holomorphic in  $\Omega_+$ . As was shown in [20] every mvf  $s \in \mathcal{S}_\kappa^{q \times p}$  admits factorizations of the form

$$(2.3) \quad s(\lambda) = b_\ell(\lambda)^{-1} s_\ell(\lambda) = s_r(\lambda) b_r(\lambda)^{-1}, \quad \lambda \in \mathfrak{h}_s^+,$$

where  $b_\ell \in \mathcal{S}^{q \times q}$ ,  $b_r \in \mathcal{S}^{p \times p}$  are Blaschke–Potapov products of degree  $\kappa$ ,  $s_\ell, s_r \in \mathcal{S}^{q \times p}$  and the factorizations (2.3) are left coprime and right coprime, respectively, i.e.

$$(2.4) \quad \text{rank} \begin{bmatrix} b_\ell(\lambda) & s_\ell(\lambda) \end{bmatrix} = q \quad (\lambda \in \Omega_+)$$

and

$$(2.5) \quad \text{rank} \begin{bmatrix} b_r(\lambda)^* & s_r(\lambda)^* \end{bmatrix} = p \quad (\lambda \in \Omega_+).$$

The following matrix identity was established in the rational case in [16], in general case see [13].

**Theorem 2.2.** *Let  $s \in \mathcal{S}_\kappa^{q \times p}$  have Kreĭn–Langer factorizations*

$$(2.6) \quad s = b_\ell^{-1} s_\ell = s_r b_r^{-1}.$$

*Then there exists a set of mvf's  $c_\ell \in H_\infty^{q \times q}$ ,  $d_\ell \in H_\infty^{p \times q}$ ,  $c_r \in H_\infty^{p \times p}$  and  $d_r \in H_\infty^{q \times p}$ , such that*

$$(2.7) \quad \begin{bmatrix} c_r & d_r \\ -s_\ell & b_\ell \end{bmatrix} \begin{bmatrix} b_r & -d_\ell \\ s_r & c_\ell \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}.$$

**2.3. The generalized Smirnov class.** Let  $\mathcal{R}^{p \times q}$  denote the class of rational  $p \times q$  mvf's and let  $\kappa \in \mathbb{N}$ . A  $p \times q$  mvf  $\varphi(z)$  is said to belong to the generalized Smirnov class  $\mathcal{N}_{+, \kappa}^{p \times q}$ , if it admits the representation

$$\varphi(z) = \varphi_0(z) + r(z), \quad \text{where} \quad \varphi_0 \in \mathcal{N}_+^{p \times q}, \quad r \in \mathcal{R}^{p \times q} \quad \text{and} \quad M_\pi(r, \Omega_+) \leq \kappa.$$

If  $\kappa = 0$ , then the class  $\mathcal{N}_{+, 0}^{p \times q}$  coincides with the Smirnov class  $\mathcal{N}_+^{p \times q}$ , defined in (2.1). The generalized Smirnov class  $\mathcal{N}_{+, \kappa}^{p \times q}$  was introduced in [23]. In [15], mvf's  $\varphi$  from  $\mathcal{N}_{+, \kappa}^{p \times q}$  were characterized by the following left coprime factorization

$$\varphi(\lambda) = b_\ell(\lambda)^{-1} \varphi_\ell(\lambda),$$

where  $b_\ell \in \mathcal{S}_{in}^{p \times p}$  is a Blaschke–Potapov product of degree  $\kappa$ ,  $\varphi_\ell \in \mathcal{N}_+^{p \times q}$  and

$$\text{rank} \begin{bmatrix} b_\ell(\lambda) & \varphi_\ell(\lambda) \end{bmatrix} = p \quad \text{for } \lambda \in \Omega_+.$$

Clearly, for  $\varphi \in \mathcal{N}_{+, \kappa}^{p \times q}$  there exists a right coprime factorization with similar properties. This implies, in particular, that the class  $\mathcal{S}_\kappa^{p \times q}$  is contained in  $\mathcal{N}_{+, \kappa}^{p \times q}$ .

**2.4. Generalized  $j_{pq}$ -inner mvf's.** Let  $j_{pq}$  be an  $m \times m$  signature matrix

$$j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad \text{where } p + q = m,$$

**Definition 2.3.** [4] An  $m \times m$  mvf  $W(\lambda) = [w_{ij}(\lambda)]_{i,j=1}^m$  that is meromorphic in  $\Omega_+$  is said to belong to the class  $\mathcal{U}_\kappa(j_{pq})$  of *generalized  $j_{pq}$ -inner mvf's*, if:

(i) the kernel

$$\mathsf{K}_\omega^W(\lambda) = \frac{j_{pq} - W(\lambda)j_{pq}W(\omega)^*}{\rho_\omega(\lambda)}$$

has  $\kappa$  negative squares in  $\mathfrak{h}_W^+ \times \mathfrak{h}_W^+$ ;

(ii)  $j_{pq} - W(\mu)j_{pq}W(\mu)^* = 0$  a.e. on  $\Omega_0$ .

As is known [4, Th.6.8.] for every  $W \in \mathcal{U}_\kappa(j_{pq})$  the block  $w_{22}(\lambda)$  is invertible for all  $\lambda \in \mathfrak{h}_W^+$  except for at most  $\kappa$  points in  $\Omega_+$ . Thus the Potapov-Ginzburg transform of  $W$

$$(2.8) \quad S(\lambda) = PG(W) := \begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ 0 & I_q \end{bmatrix} \begin{bmatrix} I_p & 0 \\ w_{21}(\lambda) & w_{22}(\lambda) \end{bmatrix}^{-1}$$

is well defined for those  $\lambda \in \mathfrak{h}_W^+$ , for which  $w_{22}(\lambda)$  is invertible. It is well known that  $S(\lambda)$  belongs to the class  $\mathcal{S}_\kappa^{m \times m}$  and  $S(\mu)$  is unitary for a.e.  $\mu \in \Omega_0$  (see [4], [13]).

**Definition 2.4.** [13] An  $m \times m$  mvf  $W \in \mathcal{U}_\kappa(j_{pq})$  is said to be in the class  $\mathcal{U}_\kappa^r(j_{pq})$ , if

$$(2.9) \quad s_{21} := -w_{22}^{-1}w_{21} \in \mathcal{S}_\kappa^{q \times p}.$$

Let  $W \in \mathcal{U}_\kappa^r(j_{pq})$  and let the Krein-Langer factorization of  $s_{21}$  be written as

$$s_{21}(\lambda) = b_\ell(\lambda)^{-1}s_\ell(\lambda) = s_r(\lambda)b_r(\lambda)^{-1} \quad (\lambda \in \mathfrak{h}_{s_{21}}^+),$$

where  $b_\ell \in \mathcal{S}_{in}^{q \times q}$ ,  $b_r \in \mathcal{S}_{in}^{p \times p}$ ,  $s_\ell, s_r \in \mathcal{S}^{q \times p}$ . Then, as was shown in [13], the mvf's  $b_\ell s_{22}$  and  $s_{11}b_r$  are holomorphic in  $\Omega_+$ , and

$$b_\ell s_{22} \in \mathcal{S}^{q \times q} \quad \text{and} \quad s_{11}b_r \in \mathcal{S}^{p \times p}.$$

**Definition 2.5.** [13] Consider inner-outer factorization of  $s_{11}b_r$  and outer-inner factorization of  $b_\ell s_{22}$

$$(2.10) \quad s_{11}b_r = b_1a_1, \quad b_\ell s_{22} = a_2b_2,$$

where  $b_1 \in \mathcal{S}_{in}^{p \times p}$ ,  $b_2 \in \mathcal{S}_{in}^{q \times q}$ ,  $a_1 \in \mathcal{S}_{out}^{p \times p}$ ,  $a_2 \in \mathcal{S}_{out}^{q \times q}$ . The pair  $\{b_1, b_2\}$  of inner factors in the factorizations (2.10) is called *the associated pair* of the mvf  $W \in \mathcal{U}_\kappa^r(j_{pq})$ .

From now onwards this pair  $\{b_1, b_2\}$  will be called also a right associated pair since it is related to the right linear fractional transformation

$$(2.11) \quad T_W[\varepsilon] := (w_{11}\varepsilon + w_{12})(w_{21}\varepsilon + w_{22})^{-1},$$

see [5], [8, 7]. Such transformations play important role in description of solutions of different interpolation problems, see [2], [5], [10], [9], [12], [14]. In the case  $\kappa = 0$  the definition of the associated pair was given in [5].

For every  $W \in \mathcal{U}_\kappa^r(j_{pq})$  and  $\varepsilon \in \mathcal{S}^{p \times q}$  the mvf  $T_W[\varepsilon]$  admits the dual representation

$$T_W[\varepsilon] = (w_{11}^\# + \varepsilon w_{12}^\#)^{-1}(w_{21}^\# + \varepsilon w_{22}^\#).$$

As was shown in [13], for  $W \in \mathcal{U}_\kappa^r(j_{pq})$  and  $c_r, d_r, c_\ell$  and  $d_\ell$  as in (2.7) the mvf

$$(2.12) \quad K^\circ := (-w_{11}d_\ell + w_{12}c_\ell)(-w_{21}d_\ell + w_{22}c_\ell)^{-1},$$

belongs to  $H_\infty^{p \times q}$ . It is clear that  $(K^\circ)^\# \in H_\infty^{q \times p}(\Omega_-)$ .

In the future we will need the following factorization of the mvf  $W \in \mathcal{U}_\kappa^r(j_{pq})$ , which was obtained in [13, Theorem 4.12]:

$$(2.13) \quad W = \Theta^\circ \Phi^\circ \quad \text{in } \Omega_+,$$

where

$$\Theta^\circ = \begin{bmatrix} b_1 & K^\circ b_2^{-1} \\ 0 & b_2^{-1} \end{bmatrix}, \quad \Phi^\circ, (\Phi^\circ)^{-1} \in \mathcal{N}_+.$$

### 3. THE TAKAGI-SARASON INTERPOLATION PROBLEM

**Problem  $\mathbf{TSP}_\kappa(b_1, b_2, K)$**  Let  $b_1 \in \mathcal{S}_{in}^{p \times p}$ ,  $b_2 \in \mathcal{S}_{in}^{q \times q}$  be inner mvf's, let  $K \in H_\infty^{p \times q}$  and let  $\kappa \in \mathbb{Z}$ . A  $p \times q$  mvf  $s$  is called a solution of the Takagi-Sarason problem  $\mathbf{TSP}_\kappa(b_1, b_2, K)$ , if  $s$  belongs to  $\mathcal{S}_{\kappa'}^{p \times q}$  for some  $\kappa' \leq \kappa$  and satisfies

$$(3.1) \quad b_1^{-1}(s - K)b_2^{-1} \in \mathcal{N}_{+, \kappa}^{p \times q}.$$

The set of solutions of the Takagi-Sarason problem will be denoted by

$$\mathcal{TS}_\kappa(b_1, b_2, K) = \bigcup_{\kappa' \leq \kappa} \{s \in \mathcal{S}_{\kappa'}^{p \times q} : b_1^{-1}(s - K)b_2^{-1} \in \mathcal{N}_{+, \kappa}^{p \times q}\}.$$

The problem  $\mathbf{TSP}_\kappa(b_1, b_2, K)$  has been studied in [11], in the rational case ( $K \in \mathcal{R}^{q \times q}$ ) the set  $\mathcal{TS}_\kappa(b_1, b_2, K) \cap \mathcal{R}^{p \times q}$  was described in [10]. In the completely indeterminate case explicit formulas for the resolvent matrix can be found in [14], [15]. In the general positive semidefinite case, the problem was solved in [17], [18].

We now recall the construction of the resolvent matrix from [15]. Let

$$\mathcal{H}(b_1) = H_2^p \oplus b_1 H_2^p, \quad \mathcal{H}_*(b_2) := (H_2^q)^\perp \oplus b_2^*(H_2^q)^\perp$$

let

$$\mathcal{H}(b_1, b_2) := \mathcal{H}(b_1) \oplus \mathcal{H}_*(b_2).$$

and let the operators  $K_{11} : H_2^q \rightarrow \mathcal{H}(b_1)$ ,  $K_{12} : \mathcal{H}_*(b_2) \rightarrow \mathcal{H}(b_1)$ ,  $K_{22} : \mathcal{H}_*(b_2) \rightarrow (H_2^p)^\perp$  and  $P : \mathcal{H}(b_1, b_2) \rightarrow \mathcal{H}(b_1, b_2)$  be defined by the formulas

$$(3.2) \quad \begin{aligned} K_{11}h_+ &= \Pi_{\mathcal{H}(b_1)}Kh_+, & h_+ &\in H_2^q, \\ K_{12}h_2 &= \Pi_{\mathcal{H}(b_1)}Kh_2, & h_2 &\in \mathcal{H}_*(b_2), \\ K_{22}h_2 &= \Pi_-Kh_2, & h_2 &\in \mathcal{H}_*(b_2), \end{aligned}$$

$$(3.3) \quad P = \begin{bmatrix} I - K_{11}K_{11}^* & -K_{12} \\ -K_{12}^* & I - K_{22}^*K_{22} \end{bmatrix}.$$

The data set  $b_1, b_2, K$  considered in [15] is subject to the following constraints:

- (H1)  $b_1 \in \mathcal{S}_{in}^{p \times p}$ ,  $b_2 \in \mathcal{S}_{in}^{q \times q}$ ,  $K \in H_\infty^{p \times q}$ .
- (H2)  $\kappa_1 = \nu_-(P) < \infty$ .
- (H3)  $0 \in \rho(P)$ .
- (H4)  $\mathfrak{h}_{b_1} \cap \mathfrak{h}_{b_2^\#} \cap \Omega_0 \neq \emptyset$ .

Define also the operator

$$(3.4) \quad F = \begin{bmatrix} I & K_{22} \\ K_{11}^* & I \end{bmatrix} : \begin{array}{c} \mathcal{H}(b_1) \\ \oplus \\ \mathcal{H}_*(b_2) \end{array} \rightarrow \begin{array}{c} b_1(H_2^p)^\perp \\ \oplus \\ b_2^*(H_2^q) \end{array} \stackrel{def}{=} \mathcal{K}.$$

As was shown in [15] for every  $h_1 \in \mathcal{H}(b_1)$  and  $h_2 \in \mathcal{H}_*(b_2)$  the vvf's  $(K_{11}^* h_1)(\lambda)$  and  $(K_{22} h_2)(\lambda)$  admit pseudocontinuations of bounded type which are holomorphic on  $\mathfrak{h}_{b_1}$  and  $\mathfrak{h}_{b_2}^\#$ , respectively. This allows to define an  $m \times m$  mvf  $\lambda \rightarrow F(\lambda)$  by

$$(3.5) \quad F(\lambda) = E(\lambda)F \quad \text{for } \lambda \in \mathfrak{h}_{b_1} \cap \mathfrak{h}_{b_2}^\#$$

where  $E(\lambda)$  is the evaluation operator

$$E(\lambda) : f \in \mathcal{K} \rightarrow f(\lambda) \in \mathbb{C}^m.$$

Let  $\mu \in \mathfrak{h}_{b_1} \cap \mathfrak{h}_{b_2}^\# \cap \Omega_0$ . Then the mvf  $W(\lambda)$  defined by

$$(3.6) \quad W(\lambda) = I - \rho_\mu(\lambda)F(\lambda)P^{-1}F(\mu)^*j_{pq} \quad \text{for } \lambda \in \mathfrak{h}_{b_1} \cap \mathfrak{h}_{b_2}^\#$$

belongs to the class  $\mathcal{U}_{\kappa_1}^r(j_{pq})$  of *generalized  $j_{pq}$ -inner* mvf's and takes values in  $L_2^{m \times m}$ . The following theorem presents a description of the set  $\mathcal{TS}_\kappa(b_1, b_2, K)$ .

**Theorem 3.1.** *Let (H1)–(H4) be in force and let  $W(\lambda)$  be the mvf, defined by (3.6). Then  $W \in \mathcal{U}_{\kappa_1}^r(j_{pq}) \cap L_2^{m \times m}$  and*

$$(1) \quad \mathcal{TS}_\kappa(b_1, b_2; K) \neq \emptyset \iff \nu_-(P) \leq \kappa.$$

$$(2) \quad \text{If } \kappa_1 = \nu_-(P) \leq \kappa, \text{ then}$$

$$(3.7) \quad \mathcal{TS}_\kappa(b_1, b_2; K) = T_W[\mathcal{S}_{\kappa-\kappa_1}^{p \times q}] := \{T_W[\varepsilon] : \varepsilon \in \mathcal{S}_{\kappa-\kappa_1}^{p \times q}\},$$

where  $T_W[\varepsilon]$  is the linear fractional transformation given by (2.11).

PROOF. The proof of this statement can be derived from the proof of Theorem 5.7 in [15]. However, we would like to present here a shorter proof based on the description of the set  $\mathcal{TS}_\kappa(b_1, b_2; K)$ , given in [14, Theorem 5.17].

As was shown in [15, see Theorem 4.2 and Corollary 4.4] the mvf  $W(z)$  belongs to the class  $\mathcal{U}_{\kappa_1}^r(j_{pq})$  of generalized  $j_{pq}$ -inner mvf's with the property (2.9) and  $\{b_1, b_2\}$  is the associated pair of  $W$ . Moreover, by construction  $W(z)$  takes values in  $L_2^{m \times m}$ . Let  $c_\ell$  and  $d_\ell$  be mvf's defined in Theorem 2.2 and let  $K^\circ$  be given by (2.12). Then  $W$  admits the factorization (2.13) (see [13, Theorem 4.12]). This proves that all the assumptions of Theorem 5.17 from [14] with  $K$  replaced by  $K^\circ$  are satisfied and by that theorem

$$(3.8) \quad \mathcal{TS}_\kappa(b_1, b_2; K^\circ) = T_W[\mathcal{S}_{\kappa-\kappa_1}^{p \times q}].$$

On the other hand it follows from [15, Theorem 4.2] that the mvf  $W$  admits the factorization

$$(3.9) \quad W = \Theta \Phi = \begin{bmatrix} b_1 & Kb_2^{-1} \\ 0 & b_2^{-1} \end{bmatrix} \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}$$

with  $\Phi, \Phi^{-1} \in \mathcal{N}_+^{m \times m}$ . Comparing (3.9) with (2.13) one obtains

$$\begin{bmatrix} I & b_1^{-1}(K - K^\circ)b_2^{-1} \\ 0 & I \end{bmatrix} = \Phi^\circ \Phi^{-1} \in \mathcal{N}_+^{m \times m}$$

and hence

$$(3.10) \quad b_1^{-1}(K - K^\circ)b_2^{-1} \in \mathcal{N}_+^{p \times q}.$$

This implies the equality  $\mathcal{TS}_\kappa(b_1, b_2; K) = \mathcal{TS}_\kappa(b_1, b_2; K^\circ)$ , that in combination with (3.8) completes the proof.  $\square$

*Remark 3.2.* Alongside with the set  $\mathcal{TS}_\kappa(b_1, b_2, K)$  consider also its subset

$$(3.11) \quad \mathcal{S}_\kappa(b_1, b_2, K) = \{s \in \mathcal{S}_\kappa^{p \times q} : b_1^{-1}(s - K)b_2^{-1} \in \mathcal{N}_{+, \kappa}^{p \times q}\}.$$

A  $p \times q$  mvf  $s$  is called a solution of the generalized Schur-Takagi problem **GSTP** $_\kappa(b_1, b_2, K)$  if  $s$  belongs to the set  $\mathcal{S}_\kappa(b_1, b_2, K)$ . A description of the set  $\mathcal{S}_\kappa(b_1, b_2, K)$  was obtained in [15, Theorem 1.2] in the form:

$$(3.12) \quad \mathcal{S}_\kappa(b_1, b_2, K) = T_W[\mathcal{S}_{\kappa - \kappa_1}^{p \times q}] \cap \mathcal{S}_\kappa^{p \times q}.$$

Notice, that the reasonings of Theorem 3.1 allows to give a shorter proof of the formula (3.12). Indeed, by [14, Theorem 5.17]

$$(3.13) \quad \mathcal{S}_\kappa(b_1, b_2; K^\circ) = T_W[\mathcal{S}_{\kappa - \kappa_1}^{p \times q}] \cap \mathcal{S}_\kappa^{p \times q}.$$

Next, it follows from (3.10) that

$$(3.14) \quad \mathcal{S}_\kappa(b_1, b_2; K) = \mathcal{S}_\kappa(b_1, b_2; K^\circ).$$

Now (3.12) is implied by (3.14) and (3.13).

#### 4. GENERALIZED $\gamma$ -GENERATING MVF'S

**Definition 4.1.** Let  $\mathfrak{M}_\kappa^r(j_{pq})$  denote the class of  $m \times m$  mvf's  $\mathfrak{A}(\mu)$  on  $\Omega_0$  of the form

$$\mathfrak{A}(\mu) = \begin{bmatrix} a_{11}(\mu) & a_{12}(\mu) \\ a_{21}(\mu) & a_{22}(\mu) \end{bmatrix},$$

with blocks  $a_{11}$  and  $a_{22}$  of size  $p \times p$  and  $q \times q$ , respectively, such that:

- (1)  $\mathfrak{A}(\mu)$  is a measurable mvf on  $\Omega_0$  and  $j_{pq}$ -unitary a.e. on  $\Omega_0$ ;
- (2) the mvf's  $a_{22}(\mu)$  and  $a_{11}(\mu)^*$  are invertible for a.e.  $\mu \in \Omega_0$  and the mvf

$$(4.1) \quad s_{21}(\mu) = -a_{22}(\mu)^{-1}a_{21}(\mu) = -a_{12}(\mu)^*(a_{11}(\mu)^*)^{-1}$$

is the boundary value of a mvf  $s_{21}(\lambda)$  that belongs to the class  $\mathcal{S}_\kappa^{q \times p}$ ;

- (3)  $a_{11}(\mu)^*$  and  $a_{22}(\mu)$ , are the boundary values of mvf's  $a_{11}^\#(\lambda)$  and  $a_{22}(\lambda)$  that are meromorphic in  $\mathbb{C}_+$  and, in addition,

$$(4.2) \quad a_1 := (a_{11}^\#)^{-1}b_r \in \mathcal{S}_{out}^{p \times p}, \quad a_2 := b_\ell a_{22}^{-1} \in \mathcal{S}_{out}^{q \times q},$$

where  $b_\ell, b_r$  are Blaschke-Potapov products of degree  $\kappa$ , determined by Krein-Langer factorizations of  $s_{21}$ .

Mvf's in the class  $\mathfrak{M}_\kappa^r(j_{pq})$  are called generalized right  $\gamma$ -generating mvf's. The class  $\mathfrak{M}^r(j_{pq}) := \mathfrak{M}_0^r(j_{pq})$  was introduced in [6], in this case conditions (2) and (3) in Definition 4.1 are simplified to:

- (2')  $s_{21} \in \mathcal{S}^{q \times p}$ ;
- (3')  $a_1 := (a_{11}^\#)^{-1} \in \mathcal{S}_{out}^{p \times p}, a_2 := a_{22}^{-1} \in \mathcal{S}_{out}^{q \times q}$ .

Mvf's from the class  $\mathfrak{M}^r(j_{pq})$  play an important role in the description of solutions of the Nehari problem and are called right  $\gamma$ -generating mvf's, [6, 7]. Mvf's in the class  $\mathfrak{M}_\kappa^r(j_{pq})$  will be called generalized right  $\gamma$ -generating mvf's.

**Definition 4.2.** [7] An ordered pair  $\{b_1, b_2\}$  of inner mvf's  $b_1 \in \mathcal{S}^{p \times p}, b_2 \in \mathcal{S}^{q \times q}$  is called a denominator of the mvf  $f \in \mathcal{N}^{p \times q}$ , if

$$b_1 f b_2 \in \mathcal{N}_+^{p \times q}.$$

The set of denominators of the mvf  $f \in \mathcal{N}^{p \times q}$  is denoted by  $\text{den}(f)$ .



**Theorem 4.3.** *Let  $\mathfrak{A} \in \Pi^{m \times m} \cap \mathfrak{M}_\kappa^r(j_{pq})$ , and let  $c_r, d_r, c_\ell$  and  $d_\ell$  be as in Theorem 2.2.*

$$(4.3) \quad f_0 = (-a_{11}d_\ell + a_{12}c_\ell)a_2.$$

*Then the mvf  $f_0$  admits the dual representation*

$$(4.4) \quad f_0 = a_1(c_r a_{21}^\# - d_r a_{22}^\#).$$

*If, in addition,  $\{b_1, b_2\} \in \text{den}(f_0)$  and*

$$(4.5) \quad W(z) = \begin{bmatrix} b_1 & 0 \\ 0 & b_2^{-1} \end{bmatrix} \mathfrak{A}(z),$$

*then  $W \in \mathcal{U}_\kappa^r(j_{pq})$  and  $\{b_1, b_2\}$  is the associated pair of  $W$ .*

*Conversely, if  $W \in \mathcal{U}_\kappa^r(j_{pq})$  and  $\{b_1, b_2\}$  is the associated pair of  $W$ , then*

$$\mathfrak{A}(z) = \begin{bmatrix} b_1^{-1} & 0 \\ 0 & b_2 \end{bmatrix} W(z) \in \Pi^{m \times m} \cap \mathfrak{M}_\kappa^r(j_{pq}) \quad \text{and} \quad \{b_1, b_2\} \in \text{den}(f_0).$$

PROOF. Let  $\mathfrak{A} \in \Pi^{m \times m} \cap \mathfrak{M}_\kappa^r(j_{pq})$ . It follows from (4.1), (4.2) and (2.3) that

$$\begin{aligned} -a_{21}d_\ell + a_{22}c_\ell &= \begin{bmatrix} a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} -d_\ell \\ c_\ell \end{bmatrix} = \begin{bmatrix} -a_{22}s_{21} & a_{22} \end{bmatrix} \begin{bmatrix} -d_\ell \\ c_\ell \end{bmatrix} \\ &= a_{22}b_\ell^{-1} \begin{bmatrix} -s_\ell & b_\ell \end{bmatrix} \begin{bmatrix} -d_\ell \\ c_\ell \end{bmatrix} \\ &= a_2^{-1}(s_\ell d_\ell + b_\ell c_\ell) = a_2^{-1}. \end{aligned}$$

Let  $f_0$  be defined by the equation (4.3). Then

$$f_0 = (-a_{11}d_\ell + a_{12}c_\ell)(-a_{21}d_\ell + a_{22}c_\ell)^{-1}.$$

The identity

$$\begin{bmatrix} c_r & -d_r \end{bmatrix} \mathfrak{A}^\# j_{pq} \mathfrak{A} \begin{bmatrix} -d_\ell \\ c_\ell \end{bmatrix} = \begin{bmatrix} c_r & -d_r \end{bmatrix} j_{pq} \begin{bmatrix} -d_\ell \\ c_\ell \end{bmatrix} = 0$$

implies that

$$(c_r a_{11}^\# - d_r a_{12}^\#)(-a_{11}d_\ell + a_{12}c_\ell) = (c_r a_{21}^\# - d_r a_{22}^\#)(-a_{21}d_\ell + a_{22}c_\ell),$$

and hence that  $f_0$  admits the dual representation

$$f_0 = (c_r a_{11}^\# - d_r a_{12}^\#)^{-1}(c_r a_{21}^\# - d_r a_{22}^\#).$$

Using the identity

$$\begin{aligned} \begin{bmatrix} c_r & -d_r \end{bmatrix} \begin{bmatrix} a_{11}^\# \\ a_{12}^\# \end{bmatrix} &= \begin{bmatrix} c_r & -d_r \end{bmatrix} \begin{bmatrix} a_{11}^\# \\ -s_{21}a_{11}^\# \end{bmatrix} \\ &= \begin{bmatrix} c_r & -d_r \end{bmatrix} \begin{bmatrix} I_p \\ -s_r b_r^{-1} \end{bmatrix} b_r a_1^{-1} = a_1^{-1} \end{aligned}$$

one obtains the equality

$$f_0 = a_1(c_r a_{21}^\# - d_r a_{22}^\#)$$

which coincides with (4.4).

Let  $\{b_1, b_2\} \in \text{den}(f_0)$ , i.e.  $b_1 f_0 b_2 \in \mathcal{N}_+^{p \times q}$ . Since  $b_1 f_0 b_2 \in L_\infty^{p \times q}$  then by Smirnov theorem

$$b_1 f_0 b_2 \in H_\infty^{p \times q}.$$

Let us find the Potapov-Ginzburg transform  $S = PG(W)$  of  $W$ , see (2.8). The formula (4.5) implies that

$$(4.6) \quad s_{21} = -w_{22}^{-1}w_{21} = -a_{22}^{-1}a_{21} = -b_\ell^{-1}s_\ell,$$

$$(4.7) \quad s_{22} = w_{22}^{-1} = a_{22}^{-1}b_2 = b_\ell^{-1}a_2b_2,$$

$$(4.8) \quad \begin{aligned} s_{11} &= w_{11}^{-*} = b_1a_1a_1^{-1}b_1^{-1}w_{11}^{-*} \\ &= b_1a_1(c_ra_{11}^* - d_ra_{12}^*)b_1^{-1}w_{11}^{-*} \\ &= b_1a_1(c_rw_{11}^* - d_rw_{12}^*)w_{11}^{-*} \\ &= b_1a_1(c_r + d_rs_{21}), \end{aligned}$$

$$(4.9) \quad \begin{aligned} s_{12} &= -w_{11}^{-*}w_{21}^* = b_1a_1(c_rw_{11}^* - d_rw_{12}^*)w_{11}^{-*}w_{21}^* \\ &= b_1a_1(c_rw_{11}^* - d_rw_{22}^* + d_rs_{22}) \\ &= b_1f_0b_2 + b_1a_1d_rs_{22}. \end{aligned}$$

The equalities (4.6)-(4.9) lead easily to the formula

$$(4.10) \quad \begin{aligned} S(z) &= \begin{bmatrix} b_1a_1c_r + b_1a_1d_rs_{21} & b_1f_0b_2 + b_1a_1d_rs_{22} \\ s_{21} & s_{22} \end{bmatrix} \\ &= \begin{bmatrix} b_1a_1c_r & b_1f_0b_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} b_1a_1d_r \\ I \end{bmatrix} \begin{bmatrix} s_{21} & s_{22} \end{bmatrix} \\ &= T(z) + \begin{bmatrix} b_1a_1d_r \\ I \end{bmatrix} b_\ell^{-1} \begin{bmatrix} -s_\ell & a_2b_2 \end{bmatrix}, \end{aligned}$$

where  $T(z) \in H_\infty^{m \times m}$ . It follows from (4.10) that

$$M_\pi(S, \Omega_+) \leq \kappa.$$

On the other hand

$$M_\pi(s_{21}, \Omega_+) = M_\pi(-b_\ell^{-1}s_\ell, \Omega_+) = \kappa,$$

and, consequently,

$$M_\pi(S, \Omega_+) = \kappa.$$

Thus,  $S \in \mathcal{S}_\kappa^{m \times m}$  and, hence,  $W \in \mathcal{U}_\kappa^r(j_{pq})$ .  $\square$

## 5. A NEHARI-TAKAGI PROBLEM

Let  $f \in L_\infty^{p \times q}$  and let  $\Gamma(f)$  be the Hankel operator associated with  $f_0$ :

$$(5.1) \quad \Gamma(f) := \Pi_- M_f|_{H_2^q},$$

where  $M_f$  denotes the operator of multiplication by  $f$ , acting from  $L_2^q$  into  $L_2^p$  and let  $\Pi_-$  denote the orthogonal projection of  $L_2^p$  onto  $(H_2^p)^\perp$ . The operator  $\Gamma(f)$  is bounded as an operator from  $H_2^q$  to  $(H_2^p)^\perp$ , moreover,

$$\|\Gamma(f)\| \leq \|f\|_{L_\infty^{p \times q}}.$$

Consider the following Nehari-Takagi problem

**Problem NTP $_\kappa(f_0)$ :** Given a mvf  $f_0 \in L_\infty^{p \times q}$ . Find  $f \in L_\infty^{p \times q}$ , such that

$$(5.2) \quad \text{rank}(\Gamma(f) - \Gamma(f_0)) \leq \kappa \quad \text{and} \quad \|f\|_\infty \leq 1.$$

In the scalar case, the problem **NTP $_\kappa(f_0)$**  has been solved by V.M. Adamyan, D.Z. Arov and M.G. Kreĭn in [1] for the case  $\kappa = 0$  and in [2] for arbitrary  $\kappa \in \mathbb{N}$ . In the matrix case a description of solutions of the problem **NTP $_0(f_0)$**  was obtained

in the completely indeterminate case by V.M. Adamyan, [3], and in the general positive-semidefinite case by A. Kheifets, [19]. The indefinite case ( $\kappa \in \mathbb{N}$ ) was treated in [11] (see also [10], where an explicit formula for the resolvent matrix was obtained in the rational case).

In what follows we confine ourselves to the case when  $\text{den}(f_0) \neq \emptyset$  and give a description of all solutions of the problem  $\mathbf{NTP}_\kappa(f_0)$ . Let us set for arbitrary  $f_0 \in L_\infty^{p \times q}$

$$\mathcal{N}_\kappa(f_0) = \{f \in L_\infty^{p \times q} : f - f_0 \in \mathcal{N}_{+, \kappa}^{p \times q}, \|f\| \leq 1\}$$

and let us denote the set of solutions of the problem  $\mathbf{NTP}_\kappa(f_0)$  by

$$\mathcal{NT}_\kappa(f_0) = \{f \in L_\infty^{p \times q} : \text{rank}(\Gamma(f) - \Gamma(f_0)) \leq \kappa \text{ and } \|f\| \leq 1\}.$$

By Kronecker Theorem ([21]), the condition  $f - f_0 \in \mathcal{N}_{+, \kappa}^{p \times q}$  is equivalent to

$$\text{rank}(\Gamma(f) - \Gamma(f_0)) = \kappa,$$

Therefore, the set  $\mathcal{NT}_\kappa(f_0)$  is represented as

$$(5.3) \quad \mathcal{NT}_\kappa(f_0) = \bigcup_{\kappa' \leq \kappa} \mathcal{N}_{\kappa'}(f_0).$$

In the following theorem relations between the set of solutions of the Nehari-Takagi problem and the set of solutions of a Takagi-Sarason problem is established in the case when  $\text{den}(f_0) \neq \emptyset$ .

**Theorem 5.1.** *Let  $f_0 \in L_\infty^{p \times q}$ ,  $\Gamma = \Gamma(f_0)$ ,  $\kappa \in \mathbb{Z}_+$ ,  $\{b_1, b_2\} \in \text{den}(f_0)$  and  $K = b_1 f_0 b_2$ . Then*

$$f \in \mathcal{N}_\kappa(f_0) \Leftrightarrow s = b_1 f b_2 \in \mathcal{TS}_\kappa(b_1, b_2, K_0).$$

PROOF. Let  $f \in \mathcal{N}_\kappa(f_0)$ . Then the mvf's  $\varphi(\mu) := f(\mu) - f_0(\mu)$ ,  $f_0(\mu)$  and  $f(\mu)$  admit meromorphic continuations  $\varphi(z)$ ,  $f_0(z)$  and  $f(z)$  on  $\Omega_+$ , such that

$$(5.4) \quad M_\pi(f - f_0, \Omega_+) = \kappa.$$

Let  $s = b_1 f b_2$  and  $K = b_1 f_0 b_2$ . Then the equality (5.4) yields

$$M_\pi(s - K, \Omega_+) \leq \kappa.$$

Since  $K \in H_\infty^{p \times q}$ , then

$$\kappa' := M_\pi(s, \Omega_+) = M_\pi(s - K, \Omega_+) \leq \kappa.$$

Taking into account that  $\|s\|_\infty = \|f\|_\infty \leq 1$ , one obtains  $s \in \mathcal{S}_{\kappa'}$ . Moreover, the condition (5.4) is equivalent to the condition (3.1), i.e.  $s \in \mathcal{TS}_\kappa(b_1, b_2, K)$ .

Conversely, if  $s \in \mathcal{S}_{\kappa'}^{p \times q}$  with  $\kappa' \leq \kappa$  and the condition (3.1) is in force, then for  $f = b_1^{-1} s b_2^{-1}$ ,  $f_0 = b_1^{-1} K b_2^{-1}$  one obtains that (5.4) holds and  $\|f\|_\infty \leq 1$ . Therefore,  $f \in \mathcal{N}_\kappa(f_0)$ .  $\square$

**Lemma 5.2.** *Let  $f_0 \in L_\infty^{p \times q}$ ,  $\Gamma = \Gamma(f_0)$ ,  $\{b_1, b_2\} \in \text{den}(f_0)$ ,  $K = b_1 f_0 b_2$  and let  $\mathbf{P}$  be the operator in  $\mathcal{H}(b_1) \oplus \mathcal{H}_*(b_2)$ , defined by formulas (3.2) and (3.3). Then*

$$\nu_-(\mathbf{P}) = \nu_-(I - \Gamma^* \Gamma).$$

Moreover, if  $\nu_-(I - \Gamma^* \Gamma) < \infty$ , then

$$0 \in \rho(\mathbf{P}) \iff 0 \in \rho(I - \Gamma^* \Gamma).$$

PROOF. Let us decompose the spaces  $H_2^q$  and  $(H_2^p)^\perp$ :

$$H_2^q = b_2(H_2^q) \oplus \mathcal{H}(b_2), \quad (H_2^p)^\perp = \mathcal{H}_*(b_1) \oplus b_1(H_2^p)^\perp$$

and let us decompose the operator  $\Gamma : H_2^q \rightarrow (H_2^p)^\perp$ , accordingly

$$(5.5) \quad \Gamma \stackrel{def}{=} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{pmatrix} : \begin{array}{c} b_2(H_2^q) \\ \oplus \\ \mathcal{H}(b_2) \end{array} \rightarrow \begin{array}{c} \mathcal{H}_*(b_1) \\ \oplus \\ b_1^*(H_2^p)^\perp \end{array},$$

where the operators

$$\Gamma_{11} : b_2(H_2^q) \rightarrow \mathcal{H}_*(b_1), \quad \Gamma_{12} : \mathcal{H}(b_2) \rightarrow \mathcal{H}_*(b_1), \quad \Gamma_{22} : \mathcal{H}(b_2) \rightarrow b_1^*(H_2^p)^\perp$$

are defined by the formulas

$$(5.6) \quad \begin{aligned} \Gamma_{11}h_+ &= \Pi_{\mathcal{H}_*(b_1)}Kh_+, & h_+ &\in b_2(H_2^q), \\ \Gamma_{12}h_2 &= \Pi_{\mathcal{H}_*(b_1)}Kh_2, & h_2 &\in \mathcal{H}(b_2), \\ \Gamma_{22}h_2 &= (b_1^*\Pi_-b_1)Kh_2, & h_2 &\in \mathcal{H}(b_2). \end{aligned}$$

It follows from (5.5), (5.6) and (3.2) that the operator  $\Gamma : H_2^q \rightarrow (H_2^p)^\perp$  and the operator

$$\mathbf{K} \stackrel{def}{=} \begin{pmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{pmatrix} : \begin{array}{c} H_2^q \\ \oplus \\ \mathcal{H}_*(b_2) \end{array} \rightarrow \begin{array}{c} \mathcal{H}(b_1) \\ \oplus \\ (H_2^p)^\perp \end{array}$$

are connected by

$$\Gamma = \mathcal{M}_{b_1^*|_{b_1(H_2^p)^\perp}} \mathbf{K} \mathcal{M}_{b_2|_{H_2^q}}.$$

and, hence, the operators  $\Gamma$  and  $\mathbf{K}$  are unitary equivalent. Now the statements are implied by [15, Lemma 5.10].  $\square$

**Theorem 5.3.** *Let  $f_0 \in L_\infty^{p \times q}$ ,  $\Gamma = \Gamma(f_0)$ ,  $\kappa \in \mathbb{Z}_+$ ,  $\{b_1, b_2\} \in \text{den}(f_0)$ ,  $K = b_1 f_0 b_2$ , let  $\mathbf{P}$  be defined by formulas (3.3), let (H1)–(H4) be in force, let the mvf  $W(z)$  be defined by (3.6) and let*

$$(5.7) \quad \mathfrak{A}(\mu) = \begin{bmatrix} b_1(\mu)^{-1} & 0 \\ 0 & b_2(\mu) \end{bmatrix} W(\mu), \quad \mu \in \mathfrak{h}_{b_1} \cap \mathfrak{h}_{b_2^\#} \cap \Omega_0.$$

Then:

- (1)  $\mathfrak{A} \in \mathfrak{M}_\kappa^r(j_{pq})$ ;
- (2)  $\mathcal{N}_\kappa(f_0) \neq \emptyset$  if and only if  $\kappa \geq \kappa_1 := \nu_-(I - \Gamma^* \Gamma)$ ;
- (3)  $\mathcal{N}_\kappa(f_0) = T_{\mathfrak{A}}[\mathcal{S}_{\kappa - \kappa_1}^{p \times q}]$ ,
- (4)  $\mathcal{NT}_\kappa(f_0) = \cup_{k=\kappa_1}^\kappa T_{\mathfrak{A}}[\mathcal{S}_{k - \kappa_1}^{p \times q}]$ .

PROOF. (1) By [15, Theorem 4.2] the rows of  $W(z)$  admit factorizations

$$\begin{bmatrix} w_{11} & w_{12} \end{bmatrix} = b_1 \begin{bmatrix} a_{11} & a_{12} \end{bmatrix},$$

$$\begin{bmatrix} w_{21} & w_{22} \end{bmatrix} = b_2^{-1} \begin{bmatrix} a_{21} & a_{22} \end{bmatrix},$$

where  $a_{11} \in (H_2^{p \times p})^\perp$ ,  $a_{12} \in (H_2^{p \times q})^\perp$ ,  $a_{21} \in H_2^{q \times p}$ ,  $a_{22} \in H_2^{q \times q}$  and

$$s_{21} = -w_{22}^{-1}w_{21} = -a_{22}^{-1}a_{21} \in \mathcal{S}_\kappa^{p \times q}.$$

If the mvf's  $b_\ell^{-1}$ ,  $s_\ell$ ,  $b_r$ ,  $s_r$  are determined by Krein-Langer factorizations of  $s_{21}$

$$s_{21} = b_\ell^{-1}s_\ell = s_rb_r^{-1},$$

then in accordance with [15, Theorem 4.3] (see (4.26), (4.27))

$$a_2 := b_\ell a_{22}^{-1} \in \mathcal{S}_{out}^{q \times q}, \quad a_1 := (a_{11}^\#)^{-1}b_r \in \mathcal{S}_{out}^{p \times p}.$$

Thus

$$\mathfrak{A}(z) = \begin{bmatrix} b_1^{-1} & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

belongs to the class  $\mathfrak{M}_\kappa^r(j_{pq})$ .

(2) By Theorem 5.1  $\mathcal{N}_\kappa(f_0)$  is nonempty if and only if  $\mathcal{TS}_\kappa(b_1, b_2, K)$  is nonempty. Therefore (2) is implied by Theorem 3.1 and Lemma 5.2.

(3) The statement (3) follows from the formula (3.7) proved in Theorem 3.1 and from the equivalence

$$f \in \mathcal{N}_\kappa(f_0) \iff b_1 f b_2 \in \mathcal{TS}_\kappa(b_1, b_2, K) T_W[\mathcal{S}_{\kappa-\kappa_1}]$$

(Theorem 5.1). This means that for every  $f \in \mathcal{N}_\kappa(f_0)$  the mvf  $s = b_1 f b_2$  belongs to  $\mathcal{TS}_\kappa(b_1, b_2, K)$  and hence it admits the representation

$$s = (w_{11}\varepsilon + w_{12})(w_{21}\varepsilon + w_{22})^{-1} = T_W[\varepsilon]$$

for some  $\varepsilon \in \mathcal{S}_{\kappa-\kappa_1}$ . Therefore, the mvf  $f = b_1^{-1} s b_2^{-1}$  can be represented as

$$f = b_1^{-1} (w_{11}\varepsilon + w_{12})(b_2 w_{21}\varepsilon + b_2 w_{22})^{-1} = T_{\mathfrak{A}}[\varepsilon].$$

(4) As follows from (2)  $\mathcal{N}_{\kappa'}(f_0) = \emptyset$  for  $\kappa' < \kappa_1$ . Therefore, (4) is implied by (5.3) and by the statement (3).  $\square$

## 6. RESOLVENT MATRIX IN THE CASE OF A RATIONAL MVF $f_0$

Assume now that  $\Omega_+ = \mathbb{D}$  and  $f_0$  is a rational mvf with a minimal realization

$$(6.1) \quad f_0(z) = C(zI_n - A)^{-1}B,$$

where  $n \in \mathbb{N}$ ,  $A \in \mathbb{C}^{n \times n}$ ,  $B \in \mathbb{C}^{n \times q}$ ,  $C \in \mathbb{C}^{p \times n}$ ,

$$(6.2) \quad \sigma(A) \subset \mathbb{D}.$$

Then the corresponding Hankel operator  $\Gamma = \Gamma(f_0) : H_2^q \rightarrow (H_2^p)^\perp$  in (5.1) has the following matrix representation  $(\gamma_{j+k-1})_{j,k=1}^\infty$  in the standard bases  $\{e^{ijt}\}_{j=0}^\infty$  and  $\{e^{-ikt}\}_{k=1}^\infty$ :

$$(\gamma_{j+k-1})_{j,k=1}^\infty = (CA^{j+k-2}B)_{j,k=1}^\infty = \Omega \Xi,$$

where  $\gamma_j$  are given by (1.1).

$$(6.3) \quad \Xi = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \quad \text{and} \quad \Omega = \begin{bmatrix} CA^0 \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Representation (6.1) is called minimal, if the dimension of the matrix  $A$  in (6.1) is minimal. As is known see [10, Thm 4.14] the representation (6.1) is minimal if and only if the pair  $(A, B)$  is controllable and the pair  $(C, A)$  is observable, i.e.

$$(6.4) \quad \text{ran } \Xi = \mathbb{C}^n \quad \text{and} \quad \ker \Omega = \{0\},$$

The controllability gramian  $P$  and the observability gramian  $Q$ , defined by

$$P = \sum_{k=0}^{\infty} A^k B B^* (A^*)^k = \Xi \Xi^*, \quad Q = \sum_{k=0}^{\infty} (A^*)^k C C^* (A)^k = \Omega^* \Omega,$$

are solutions to the following Lyapunov-Stein equations

$$(6.5) \quad P - A P A^* = B B^*, \quad Q - A^* Q A = C^* C.$$

As was shown in [14, Remark 4.2] a denominator of the mvf  $f_0(z)$  may be selected as  $(I_p, b_2)$ , where

$$(6.6) \quad b_2(z) = I_q - (1 - z)B^*(I_n - zA^*)^{-1}P^{-1}(I_n - A)^{-1}B$$

Straightforward calculations show that

$$(6.7) \quad (zI_n - A)^{-1}Bb_2(z) = P(I_n - A^*)(I_n - zA^*)^{-1}P^{-1}(I_n - A)^{-1}B.$$

Since the mvf  $b_2(z)$  is inner, then  $b_2(z)^{-1} = b_2(\frac{1}{z})^*$ , and hence

$$(6.8) \quad b_2(z)^{-1} = I_q + (1 - z)B^*(I_n - A^*)^{-1}P^{-1}(zI_n - A)^{-1}B.$$

**Proposition 6.1.** *Let  $f_0(z)$  be a mvf of the form (6.1), where  $A \in \mathbb{C}^{n \times n}$ ,  $B \in \mathbb{C}^{n \times q}$ ,  $C \in \mathbb{C}^{p \times n}$  satisfy (6.2) and (6.4) and let*

$$(6.9) \quad M = \begin{bmatrix} -A & 0 \\ 0 & I_n \end{bmatrix}, \quad N = \begin{bmatrix} -I_n & 0 \\ 0 & A^* \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -Q & I_n \\ I_n & -P \end{bmatrix},$$

$$(6.10) \quad G(z) = \begin{bmatrix} C & 0 \\ 0 & B^* \end{bmatrix} (M - zN)^{-1}.$$

Assume that  $1 \notin \sigma(PQ)$ . Then:

- (1)  $\mathcal{N}_\kappa(f_0) \neq \emptyset$  if and only if  $\kappa_1 := \nu_-(I - PQ) \leq \kappa$ ;
- (2) If (1) holds then the matrix  $\Lambda$  is invertible and  $\mathcal{N}_\kappa(f_0) = T_{\mathfrak{A}}[\mathcal{S}_{\kappa - \kappa_1}]$ , where

$$(6.11) \quad \mathfrak{A}(\mu) = I_m - (1 - \mu)G(\mu)\Lambda^{-1}G(1)^*j_{pq};$$

- (3) The mvf  $\mathfrak{A}(\mu)$  is a generalized right  $\gamma$ -generating mvf of the class  $\mathfrak{M}_{\kappa_1}^r(j_{pq})$ .

The statements (1), (2) of Proposition 6.1 and the formula (6.11) for the resolvent matrix  $\mathfrak{A}(\mu)$  are well known from [10, Theorem 20.5.1]. We will deduce now the formula (6.11) from the general formula (3.6) for the resolvent matrix of the problem  $\mathbf{TSP}_\kappa(I_p, b_2, K)$  with

$$(6.12) \quad K(z) = f_0(z)b_2(z) = C(zI_n - A)^{-1}Bb_2(z).$$

PROOF. (1) By Theorem 5.1  $f \in \mathcal{N}_\kappa(f_0)$  if and only if  $s = fb_2 \in \mathcal{TS}_\kappa(I_p, b_2, K)$ . Alongside with the problem  $\mathbf{TSP}_\kappa(I_p, b_2, K)$  consider the problem  $\mathbf{GSTP}_\kappa(I_p, b_2, K)$  (see Remark 3.2). As is known [14, Theorem 5.17] these problems have the same resolvent matrix. Assume that  $s \in \mathcal{S}_\kappa(I_p, b_2, K)$ , see (3.11). The conditions  $s \in \mathcal{S}_\kappa^{p \times q}$  and  $(s - K)b_2^{-1} \in \mathcal{N}_{+, \kappa}$  are equivalent to the equalities

$$M_\pi(s, \Omega_+) = M_\pi((s - K)b_2^{-1}, \Omega_+) = \kappa.$$

By the cancellation lemma from [14, Lemma 5.5]

$$(6.13) \quad M_\pi((s_\ell - b_\ell K)b_2^{-1}, \Omega_+) = M_\pi(s_\ell, \Omega_+) = 0.$$

By (6.8) and (6.12) the expression  $(s_\ell - b_\ell K)b_2^{-1}$  takes the form

$$s_\ell(I_q + (1 - z)B^*(I_n - A^*)^{-1}P^{-1}(zI_n - A)^{-1}B) - b_\ell C(zI_n - A)^{-1}B.$$

Since  $(1 - z)(zI_n - A)^{-1} = -I_n + (I_n - A)(zI_n - A)^{-1}$ , the condition (6.13) can be rewritten as

$$(6.14) \quad \{s_\ell B^*(I_n - A^*)^{-1}P^{-1}(I_n - A) - b_\ell C\}(zI_n - A)^{-1}B \in \mathcal{N}_+.$$

Since the pair  $(A, B)$  is controllable, then (6.14) is equivalent to

$$(6.15) \quad \begin{bmatrix} b_\ell & -s_\ell \end{bmatrix} \begin{bmatrix} C \\ B^*(I_n - A^*)^{-1}P^{-1}(I_n - A) \end{bmatrix} (zI_n - A)^{-1} \in \mathcal{N}_+.$$

Thus, the condition (6.15) can be rewritten as

$$(6.16) \quad \begin{bmatrix} b_\ell & -s_\ell \end{bmatrix} F \in \mathcal{N}_+,$$

where

$$(6.17) \quad F(z) = \tilde{C}(A - zI_n)^{-1}, \quad \tilde{C} = \begin{bmatrix} C \\ B^*(I_n - A^*)^{-1}P^{-1}(I_n - A) \end{bmatrix}.$$

Thus, the problem  $\mathbf{GSTP}_\kappa(I_p, b_2, K)$  is equivalent to the one-sided interpolation problem (6.16) considered in [14]. As was shown in [14, (1.14)] the Pick matrix  $\tilde{P}$ , corresponding to the problem (6.16) is the unique solution of the Lyapunov-Stein equation

$$(6.18) \quad A^*\tilde{P}A - \tilde{P} = \tilde{C}^*j_{pq}\tilde{C}$$

and the problem (6.16) is solvable if and only if

$$\kappa_1 := \nu_-(\tilde{P}) \leq \kappa.$$

Since by (6.5)

$$(6.19) \quad \begin{aligned} \tilde{C}^*j_{pq}\tilde{C} &= C^*C - (I_n - A^*)P^{-1}(I_n - A)^{-1}BB^*(I_n - A^*)^{-1}P^{-1}(I_n - A) \\ &= C^*C - (I_n - A^*)P^{-1} - A^*P^{-1}(I_n - A) \\ &= (Q - P^{-1}) - A^*(Q - P^{-1})A, \end{aligned}$$

then

$$(6.20) \quad \tilde{P} = P^{-1} - Q = P^{-1/2}(I - P^{1/2}QP^{1/2})P^{-1/2}.$$

Notice, that in (6.19) we use the equality

$$-(I_n - A)^{-1}BB^*(I_n - A^*)^{-1} = -(I_n - A)^{-1}P - PA^*(I_n - A^*)^{-1},$$

It follows from (6.20) and Theorem 3.1 that  $\mathcal{TS}_\kappa(I_p, b_2, K) \neq \emptyset$  if and only if

$$\kappa_1 := \nu_-(I - P^{1/2}QP^{1/2}) \leq \kappa.$$

Now it remains to note that  $\sigma(I - P^{1/2}QP^{1/2}) = \sigma(I - PQ)$ . In view of Theorem 5.1 this proves (1).

(2) By [14, Theorem 3.1 and Theorem 5.17] the resolvent matrix  $\widetilde{W}(z)$ , which describes the set  $\mathcal{TS}_\kappa(I_p, b_2, K_0)$  via the formula (3.8) takes the form

$$\widetilde{W}(z) = I_m - (1 - z)F(z)\tilde{P}^{-1}F(1)^*j_{pq},$$

where  $\tilde{P}$  is given by (6.18). As was shown in [14, Lemma 4.8]  $\widetilde{W} \in \mathcal{U}_\kappa^r(j_{pq})$ . Let us set

$$(6.21) \quad \tilde{\mathfrak{A}}(\mu) := \begin{bmatrix} I_p & 0 \\ 0 & b_2(\mu) \end{bmatrix} W(\mu)$$

and show that the mvf

$$(6.22) \quad \tilde{\mathfrak{A}}(\mu) := \begin{bmatrix} I_p & 0 \\ 0 & b_2(\mu) \end{bmatrix} + (\mu - 1) \begin{bmatrix} I_p & 0 \\ 0 & b_2(\mu) \end{bmatrix} F(\mu)\tilde{P}^{-1}F(1)^*j_{pq}$$

coincides with the mvf  $\mathfrak{A}$  from (6.11). It follows from (6.7) that

$$b_2(\mu)^{-1}B^*(I_n - \mu A^*)^{-1} = B^*(I_n - A^*)^{-1}P^{-1}(\mu I_n - A)^{-1}(I_n - A)P.$$

and hence

$$b_2(\mu)B^*(I_n - A^*)^{-1}P^{-1}(\mu I_n - A)^{-1}(I_n - A) = B^*(I_n - \mu A^*)^{-1}P^{-1}.$$

In view of (6.6), (6.17), (6.9) and (6.10) this implies

$$(6.23) \quad \begin{bmatrix} I_p & 0 \\ 0 & b_2(\mu) \end{bmatrix} F(\mu) = \begin{bmatrix} C(\mu I_n - A)^{-1} \\ B^*(I_n - \mu A^*)^{-1} P^{-1} \end{bmatrix} = G(\mu) \begin{bmatrix} I_p \\ P^{-1} \end{bmatrix}.$$

Next, in view of (6.17) and (6.6)

$$(6.24) \quad F(1)^* = [(I_n - A^*)^{-1} C^* \quad P^{-1}(I_n - A)^{-1} B] = [I_n \quad P^{-1}] G(1)^*,$$

$$(6.25) \quad \begin{bmatrix} I_p & 0 \\ 0 & b_2(\mu) \end{bmatrix} = I_m - (1 - \mu) \begin{bmatrix} 0 & 0 \\ 0 & B^*(I_n - \mu A^*)^{-1} P^{-1}(I_n - A)^{-1} B \end{bmatrix} \\ = I_m - (1 - \mu) G(\mu) \begin{bmatrix} 0 & 0 \\ 0 & -P^{-1} \end{bmatrix} G(1)^* j_{pq}.$$

Substituting (6.23), (6.24) and (6.25) into (6.22) one obtains

$$\tilde{\mathfrak{A}}(\mu) = I_m - (1 - \mu) G(\mu) \begin{bmatrix} \tilde{P}^{-1} & \tilde{P}^{-1} P^{-1} \\ P^{-1} \tilde{P}^{-1} & -P^{-1} + P^{-1} \tilde{P}^{-1} P^{-1} \end{bmatrix} G(1)^* j_{pq}.$$

In view of the equality

$$\begin{bmatrix} \tilde{P}^{-1} & \tilde{P}^{-1} P^{-1} \\ P^{-1} \tilde{P}^{-1} & -P^{-1} + P^{-1} \tilde{P}^{-1} P^{-1} \end{bmatrix} = \Lambda^{-1}$$

this proves the formula (6.11).

By [14, Theorem 3.1 and Theorem 5.17] and Theorem 3.1 the set  $\mathcal{TS}_\kappa(I_p, b_2, K_0)$  is described by the formula

$$\mathcal{TS}_\kappa(b_1, b_2; K) = T_W[\mathcal{S}_{\kappa-\kappa_1}^{p \times q}] = \{T_W[\varepsilon] : \varepsilon \in \mathcal{S}_{\kappa-\kappa_1}^{p \times q}\}.$$

Therefore, the statement (3) is implied by Theorem 5.3 (3).

(3) By Theorem 5.3 the mvf  $\tilde{\mathfrak{A}}(\mu)$  defined by (6.21) belongs to the class  $\mathfrak{M}_{\kappa_1}^r(j_{pq})$ .

□

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